Summary: conditional expectation
exercises

Recall:

- Filtration
  - → definition ($\sigma$-algebra)
  - → Relation between filtration and information

11. Show that if $\{\mathcal{F}_i, i \in J\}$ is any collection of $\sigma$-algebras on the same set $\Omega$, then their intersection, $\bigcap_{i\in J} \mathcal{F}_i$, is also a $\sigma$-algebra on $\Omega$.

[Recall from past courses:
- probability distribution; density function
- Expected value of a Random variable
- Properties of a normal distribution ]

Each $\mathcal{F}_i$ is a $\sigma$-algebra, $i = 1, 2, .., n$

And we want to prove that

$$\mathcal{G} = \bigcap_{i=1}^{n} \mathcal{F}_i \text{ is also a } \sigma\text{-algebra}$$

We need to check that:

- $\phi \in \mathcal{G}$

  $\mathcal{F}_i$ is a $\sigma$-algebra $\Rightarrow \phi \in \mathcal{F}_i, \ \forall i \in \{1, .., n\}$

  $\Rightarrow \phi \in \bigcap_{i=1}^{n} \mathcal{F}_i$ ✓

- $A \in \mathcal{G} \Rightarrow \bar{A} \in \mathcal{G}$

  if $A \in \mathcal{G} \Rightarrow A \in \mathcal{F}_i, \ \forall i \in \{1, 2, .., n\}$

  As $\mathcal{F}_i$ is a $\sigma$-algebra $\Rightarrow \bar{A} \in \mathcal{F}_i, \forall i$

  $\Rightarrow \bar{A} \in \bigcap \mathcal{F}_i$ ✓

- $A_i \in \mathcal{G} \Rightarrow \bigcup A_i \in \mathcal{G}$

For ease of presentation, we prove that:

$A$ and $B \in \mathcal{G} \Rightarrow A \cup B \in \mathcal{G}$

$$\left.\begin{array}{l} A \in \mathcal{G} \Rightarrow A \in \mathcal{F}_i, \ \forall i \\ B \in \mathcal{G} \Rightarrow B \in \mathcal{F}_i, \ \forall i \end{array}\right\} \Rightarrow \text{As } \mathcal{F}_i \text{ is a } \sigma\text{-algebra,}$$

Then $A \cup B \in \mathcal{F}_i, \ \forall i \Rightarrow A \cup B \in \mathcal{G}$ ✓

So intersections of σ-algebras is still a σ-algebra.
What about unions of σ-algebras?

$\Omega = \{1, 2, 3, 4\}$ $\quad \mathcal{F}_1 = \{\phi, \Omega, \{1,2\}, \{3,4\}\}$

$\mathcal{F}_2 = \{\phi, \Omega, \{4\}, \{1,2,3\}\}$

$\mathcal{F}_1 \cup \mathcal{F}_2 = \{\phi, \Omega, \{1,2\}, \{3,4\}, \{4\}, \{1,2,3\}\}$

$\underbrace{\{1,2\} \cup \{4\}}_{\{1,2,4\}} \notin \mathcal{F}_1 \cup \mathcal{F}_2$

Information and filtrations/σ-algebras are close to each others. How do we use this information?

$\Downarrow$

Conditional expectations

Example

| Y\X | 0 | 1 | 2 | |
|---|---|---|---|---|
| 0 | 0.1 | 0.1 | 0.2 | 0.4 |
| 1 | 0.05 | 0.05 | 0.1 | 0.2 |
| 2 | 0.2 | 0.15 | 0.05 | 0.4 |
| | 0.35 | 0.3 | 0.35 | |

$P(X=2, Y=1)$

$$E[X] = \sum_{x=-\infty}^{+\infty} x\, P(X=x) =$$

$$= 0 \times 0.35 + 1 \times 0.3 + 2 \times 0.35$$

$$= 1.0$$

$$\underbrace{E[X \mid Y=2]}_{} = \sum_{x=0}^{2} x \frac{P(X=x, Y=2)}{P(Y=2)}$$

we add information about $Y$: we know that $Y$ is equal to 2

$$E[X \mid Y] = \begin{cases} E[X \mid Y=0] & \text{with prob. } 0.4 \\ E[X \mid Y=1] & \text{with prob. } 0.2 \\ E[X \mid Y=2] & \text{with prob. } 0.4 \end{cases}$$

This example shows:

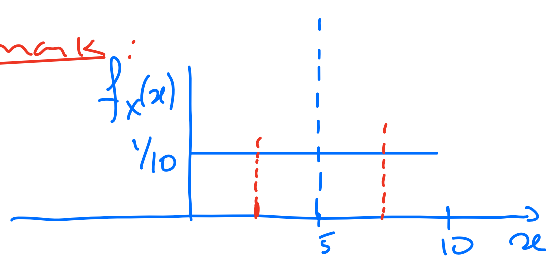
$E[X \mid Y=2]$ is a deterministic value ($Y$ is completely known)

$E[X \mid Y]$ is a random variable, that depends on $Y$.

In general, given 2 random variables $X$ and $Y$ [defined on a certain probability space], $E[X \mid Y=y]$ will be a deterministic value

$$\boxed{g(Y) \equiv E[X \mid Y] \text{ will be a random variable}}$$

Remark:




$X \sim \text{Unif}(0, 10)$

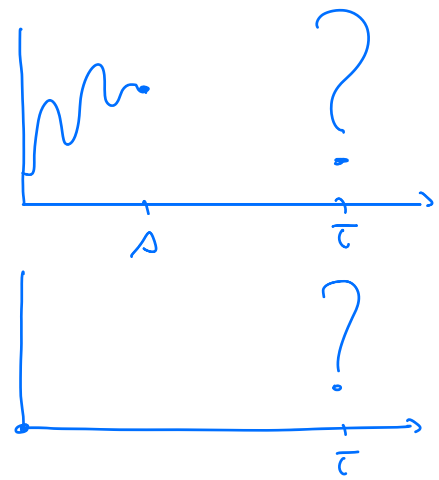
$E[X] = 5$

$E[X \mid X < 5]$

$E[X \mid X > 5]$

Here, in the course, we want to use

time $= s$ $\quad E[S_T \mid \mathcal{F}_s] = ?$




"Elementary stochastic calculus, with Finance in view," from Thomas Mikosh

The definition of conditional expectation is challenging and we will skip it.
Moreover, we don't need to compute conditional expectations.
What is relevant in the course are the properties of conditional expectation!

## Properties of conditional expectation

Let $\mathcal{F}$ be two $\sigma$-algebras, with $\mathcal{G} \subseteq \mathcal{F}$ and let $X$ be a random variable measurable with respect to $\mathcal{G}$.
(measurable = adapted)

- $E[aX \mid \mathcal{F}] = a\,E(X \mid \mathcal{F}]$ $\quad \forall a$
  $\left( E[aX] = a\,E[X] \right)$

- $E[a \mid \mathcal{F}] = a$

- $E[X \mid \mathcal{G}] = X$

- Let $Y$ be another random variable such that

$Y$ is $\mathcal{F}$-measurable.

## Properties of conditional expectation

Let $\mathcal{F}$ be a $\sigma$-algebra and let $X$ be a random variable.

- $E[aX \mid \mathcal{F}] = a\,E[X \mid \mathcal{F}] \quad \forall a$

  $\left( E[aX] = a\,E[X] \right)$

- $E[a \mid \mathcal{F}] = a$

- If $X$ is $\mathcal{F}$-measurable, then
  $E[X \mid \mathcal{F}] = X$

- Let $Y$ be another variable, $\mathcal{F}$-measurable. Then
  $$E[XY \mid \mathcal{F}] = Y\,E[X \mid \mathcal{F}]$$
  (with $X$ not necessarily being $\mathcal{F}$-measurable)

- Tower property: Let $\mathcal{F}$ and $\mathcal{G}$ be two $\sigma$-algebras with $\mathcal{G} \subseteq \mathcal{F}$. Then
  $$E\big[E[X \mid \mathcal{F}] \mid \mathcal{G}\big] = E\big[E[X \mid \mathcal{G}] \mid \mathcal{F}\big] = E[X \mid \mathcal{G}]$$

  $$\left( E[E[X \mid Y]] = E[X] \right)$$

Note: $E[X \mid \mathcal{F}]$ is a random variable, is measurable with respect to $\mathcal{F}$.

$\left( E(X) = \sum x\,P(X = x) \qquad E[X] = \int x f_X(x)\,dx \right)$

6. Consider a sequence $X = \{X_n, n \in \mathbb{N}\}$, and let $\sigma(X_1, \ldots, X_n) = \mathcal{F}_n$, such that $X$ is a submartingale, i.e., $\mathbb{E}[X_{n+1}|\mathcal{F}_n] \geq X_n$. Moreover, let $A_0 = 0$ and:

$$A_n = \sum_{k=1}^{n} (\mathbb{E}[X_k|\mathcal{F}_{k-1}] - X_{k-1}) \quad M_n = \sum_{k=1}^{n} (X_k - \mathbb{E}[X_k|\mathcal{F}_{k-1}])$$

a) Prove that $\{M_n, n \in \mathbb{N}\}$ is a martingale.



b) Prove that $A_{n+1} \geq A_n$, for all $n$. The r.v. $A_n$ is $\mathcal{F}_{n-1}$-measurable? And $\mathcal{F}_n$-measurable?

8. Let $X = \{X_n, n \in \mathbb{N}\}$ be a martingale on $(\Omega, \mathcal{F}, P)$, and assume that $\mathcal{G} = \{G_n, n \in \mathbb{N}\}$ is a filtration contained in $\mathcal{F}$, and such that $X$ is $\mathcal{G}$-adapted. Then show that $X$ is also a $(\mathcal{G}, P)$-martingale.